Controlling continuous chaotic dynamics by periodic proportional pulses

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(Received 2 September 1997)

It has been shown that proportional pulses $X(t)\rightarrow kX(t)$, applied at regular time intervals to an unknown chaotic dynamic, may stabilize the dynamic at a periodic orbit. Given an integer p, we showed where proportional pulses can stabilize the dynamic $X'(t)=F(X(t))$ at a periodic orbit of period p and how to calculate the corresponding factor k. The existence of a Poincaré section is assumed. [S1063-651X(98)09001-1]

PACS number(s): 05.45.+b

In the last decade, a large number of reports have investigated the problem of controlling chaos. The aim was to stabilize the dynamic at a periodic orbit. Ott, Grebogy, and York [1] performed small modifications of a system's parameter to stabilize a chaotic orbit. Dressler and Nitsche [2] applied the method to delay coordinates. Hartley and Mossayebi [3], and Pyragas [4] used feedback and delay feedback to master chaos. Braiman [5] tamed chaos by introducing small periodic perturbations into a system parameter. Interaction between a system and its subsystems can synchronize chaotic dynamics [6]. We have used a Kalman filter to control chaos in the presence of large dynamical noise [7]. Maneuvers to control chaos have been tested in different areas, including laboratory physics [8–10], cardiology [11], and biochemistry [12].

An interesting maneuver to control chaos was introduced by Matias and Güémez [13,14]. In the case of a continuous dynamic, the authors performed instantaneous pulses on the system variables, $X(t)$, at a period τ, in the form

$$X(t)\rightarrow kX(t) \quad (\text{with } t=p\tau), \tag{1}$$

where k is a constant. Matias and Güémez have given several examples in which Eq. (1) actually stabilizes chaotic dynamics at a periodic orbit. If the dynamic is unknown, one can only test the control with several values of τ and k. However, when the dynamic is known, as in the examples shown in their work, the authors did not give the method to calculate the constants τ and k, in order to apply the proportional control.

In the present work, we raised the following general question: Given a dynamic

$$X'(t)=F(X(t)) \tag{2}$$

an arbitrary point M in the phase space, and an arbitrary integer p, can one, by using proportional control, stabilize the dynamic at a periodic orbit of given period p and crossing the given point M? In the case where the dynamic possesses a Poincaré section, we could give an answer (yes/no) to this question and when the answer is yes, show how to calculate the factor k and where to perform the control to achieve the stabilization.

We suppose that a Poincaré section of this dynamic is available. The generic points of this section will be denoted by $X(i)$ where the index i designates the successive visits of the orbit to the section. We assumed that in the Poincaré plane referred to coordinates (x,y), the section $(x(i),y(i))$ could be modelized in the form

$$x(i+1)=f(x(i)). \tag{3}$$

This assumption means that, for a given trajectory, the value of x at the instant when the trajectory crosses the section uniquely specifies the components of $X(t)$, i.e., that two constants of motion exist. In a general term, assumption (3) requires that $n-2$ constants of motion exist, when the dynamic is of dimension n.

Suppose now that we kick at the orbit when it visits the Poincaré section, by multiplying $x(i)$ by a factor k, once every p visits. Let

$$g(x)=kf^{(p)}(x), \tag{4}$$

where $f^{(p)}$ is the p-times composition of the map f with itself. A fixed point x_s of g is any solution of the equation

$$kf^{(p)}(x_s)=x_s \tag{5}$$

and this fixed point is locally stable if

$$-1<kf^{(p)'}(x_s)<1, \tag{6}$$

where the prime designates the derivative of the composite function. A stable fixed point of g defines a close orbit of the initial dynamic kicked in the Poincaré section by the control procedure.

We suppose that the original map is chaotic and wish to control it to a stable periodic orbit of period p, by kicking at its orbit once every p visits of the orbit to the Poincaré section.

To stabilize the dynamic, it is sufficient to find a point x_s on the Poincaré section and a factor k satisfying Eq. (5) and inequalities Eq. (6). Taking k from Eq. (5) and defining

$$C_p(x)=\frac{xf^{(p)'}(x)}{f^{(p)}(x)} \tag{7}$$

Eq. (6) becomes

$$-1<C_p(x)<1. \tag{8}$$

Inequalities (8) are the key of the method: if a point x_s satisfies Eq. (8), and only in that case, then with the kicking

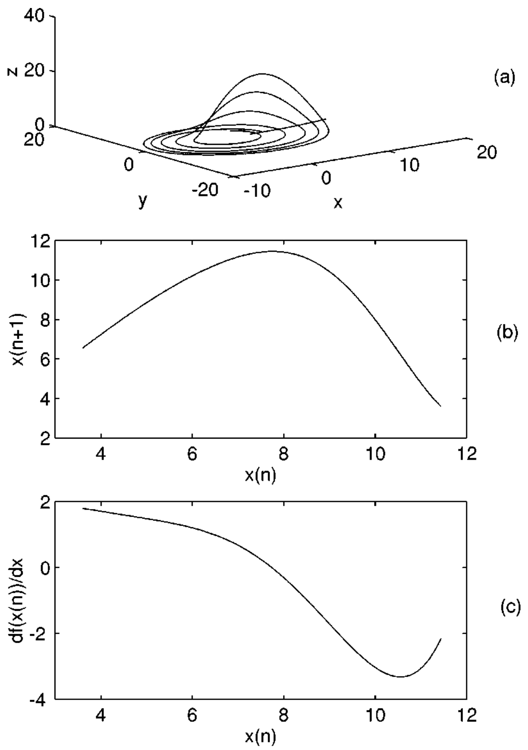

FIG. 1. (a) Part of an orbit of the Rössler dynamic. The transversal (a) part of an orbit of the Rössler dynamic. The transversal line is the Poincaré section (points where the x variable reaches a maximal level). (b) Recurrent map of the Poincaré section. (c) Derivative of the recurrent map [derivative of the function shown in (b)].

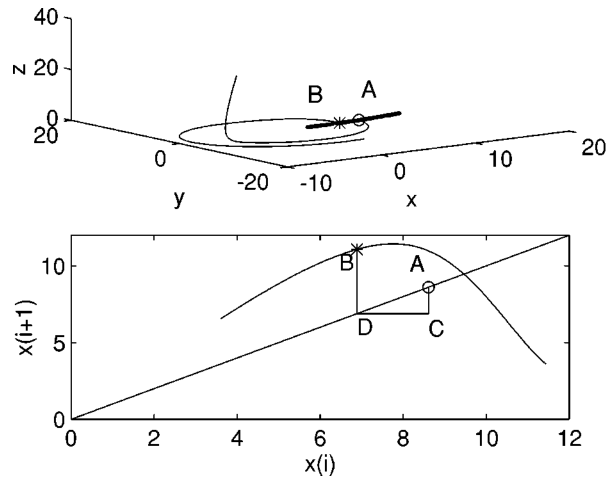

FIG. 2. Control procedure: (a) The orbit arrives at the section at point A, and kicked by the control procedure to point B. (b) Web diagram of the kicking procedure: the x coordinate of point A is multiplied by the kicking factor k (0.8 in this case) to get the point C, from which one joins the diagonal at D and the recurrent curve at B. This gives the x coordinate of B. The y and z coordinates of B are obtained by interpolation from coordinates of the Poincaré section.

factor k defined by Eq. (5), the control procedure will stabilize the dynamic at a periodic orbit of period p, passing through the point x_s.

In practice, controlling chaos is interesting only when we can stabilize the trajectory at a periodic orbit of ''low'' periods, say, $p=1-5$. In that case, we may proceed as follows. We generate a trajectory starting from an arbitrary point. At each visit of the orbit to the Poincaré section, we test inequalities (8) for $p=1$, 2, 3, 4, and 5. If Eq. (8) is satisfied for such a period p, we calculate the factor k using Eq. (5) and operate the control immediately. The trajectory will be stabilized at a periodic orbit of period p crossing the given point. By ergodicity, our trajectory will visit almost all points of the section, therefore our method can explore the possibility to stabilize the dynamic at almost all points of the Poincaré section.

Consider, as an example, the Rössler dynamic [15]

$$x'(t)=-y(t)-z(t),$$

$$y'(t)=x(t)+0.2y(t),$$

$$z'(t)=0.2+z(t)[x(t)-5.7].$$

This map has a well-known Poincaré section defined by the points where $y(t)+z(t)$ crosses zero from negative to positive values, or equivalently when $x(t)$ reaches a local maximum. Figure 1(a) shows a portion of the Rössler trajectory, and its successive visits to the Poincaré section (the Poincaré section is the transversal line cutting the orbit at points where

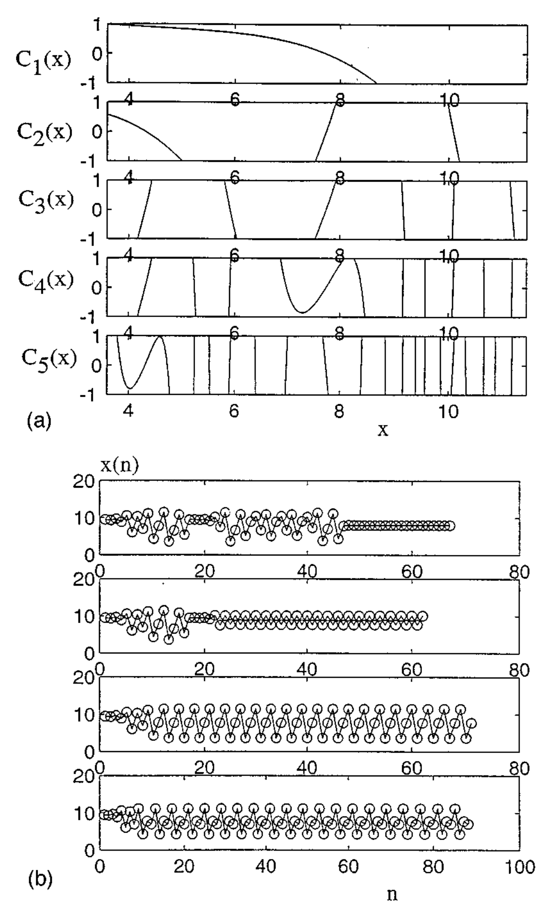

FIG. 3. (a) The function $C_p(x)$ in the Poincaré section and for $p=1$ to 5. Values of the function in the range -1, 1 are shown. If an x value is such that $-1<C_p(x)<1$, then the kicking procedure can stabilize the trajectory at a periodic orbit crossing x. (b) Four examples of stabilizing the orbit in the Poincaré section.

$x(t)$ is maximal). Figure 1(b) shows the relationship relating successive visits $x(n)$ and $x(n+1)$ of the orbit to the Poincaré section. Figure 1(c) shows the derivative of the recurrent map relating $x(n)$ to $x(n+1)$. The derivative was calculated numerically.

Figure 2 shows how to perform the kicking control. The orbit visits the Poincaré section at the point A in Fig. 2(a). The control procedure will move it to the point B. To calculate the position of B, we go to the web diagram in Fig. 2(b). The value of x at A is multiplied by the kicking factor k (0.8 in this example) to get the point C, from there we joint the diagonal at D and then the recurrent map at B. Once the x coordinate of B is known, we look at the table of points $(x(i),y(i),z(i))$ of the Poincaré section. By interpolation, we calculate the y and z coordinates corresponding to the x coordinate of B. This gives the position of point B in Fig. 2(a), from there we free the orbit and it continues its way following the Rössler dynamic.

To control the dynamic, it is sufficient to look for points in the Poincaré section that satisfy inequalities (8). Figure 3(a) shows the function $C_p(x)$ for $p=1,\ 2,\ 3,\ 4$, and 5, and for all x in the Poincaré section. We drew only $C_p(x)$ values within -1 and 1. The figure shows that for $p=1$, we can stabilize the orbit at every point up to about 8.6 and only at these points. For $p=2$, we can stabilize the orbit at 3 ranges of x values, shown in the figure. There are 6 acceptable ranges of x values for period 3, 10 narrow ranges of x values for period 4, and 18 intervals of acceptable x values for period 5. Figure 3(b) shows 4 examples of stabilizing the recurrent map at period 1, 2, 3, and 4. Figure 4 shows the stabilized orbits, of periods 2, 3, 4, and 5, controlled by the kicking method.

The above method explores the possibility to control the dynamic at a periodic orbit crossing any point on the Poincaré section. If one wishes to stabilize the orbit at a cycle passing through a point not on the section, it is sufficient to follow the trajectory from that point until it arrives at a point on the Poincaré section and then apply there the control procedure. A trajectory issued from any point will reach the Poincaré section at a certain moment. Therefore, our method explores in fact the possibility to stabilize the orbit through any point in the attractor.

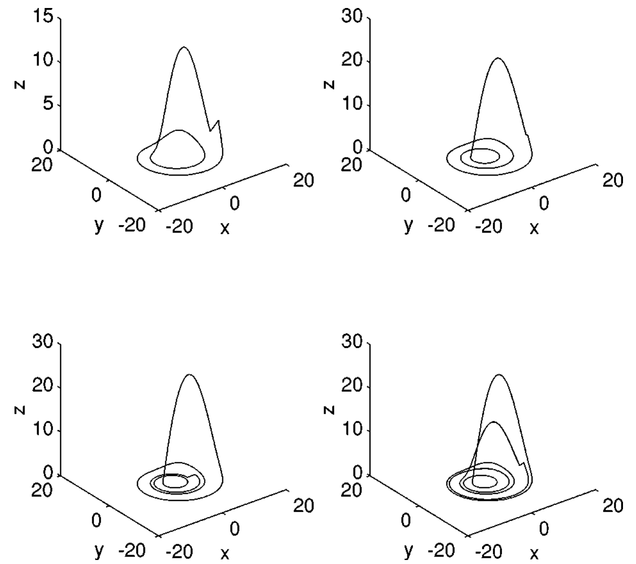

FIG. 4. Examples of stabilizing the Rössler's orbit at periods 2, 3, 4, and 5, using the kicking procedure.

To conclude, we have presented an analysis of the effect of regularly kicking at the trajectory of a continuous chaotic dynamic when the orbit reaches a given Poincaré section. Here kicking means multiplying a variable of the section by a constant. We have shown that such a procedure can stabilize the dynamics at a large number of different points on the section. For each p, we have shown how to calculate the kicking factor k, which may stabilize the dynamic.

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